

# Phase Flows and Vector Hamiltonians

V.N.Dumachev<sup>1</sup>

Voronezh Institute of the MVD of the Russia

We present a generalization of the Nambu mechanics on the base of Liouville's theorem. We prove that the Poisson structure of an  $n$ -dimensional multisymplectic phase space is induced by  $(n-1)$ -Hamiltonian  $k$ -vector field each of which requires introduction of  $k$ -Hamiltonians.

**Keywords:** Liouville theorem, Hamiltonian vector fields.

**1.** Consider a system of differential equations as a submanifold  $\Sigma$  in a jet bundle  $J^n(\pi): E \rightarrow M$ , defined by equations ([1], P.10)

$$g(t, x_0, x_1, \dots, x_n) = 0,$$

where  $t \in M \subset R$ ,  $u = x_0 \in U \subset R$ ,  $x_i \in J^i(\pi) \subset R^n$ ,  $E = M \times U$ . For equations  $\Sigma \subset J^n(\pi)$  which admit a Poisson structure with bracket

$$\{H, G\} = X_H]dG = L_{X_H}G \quad (1)$$

the Cartan distribution can be given as follows:

$$\theta_i = dx_i - \{H, x_i\}dt. \quad (2)$$

Here  $L_{X_H}$  is the Lie derivative with respect to  $X_H \in \Lambda^1$ ,  $\Lambda^n$  is the exterior graded algebra of  $k$ -polyvector fields,  $H = H(\mathbf{x})$  is an unknown function.

**2.** The classical symplectic mechanics on  $J^2(\pi)$  admits a Hamiltonian structure induced by a vector field  $X_H^1$ . A vector field  $X_H^1$  on a symplectic manifold  $(M, \omega)$  is called Hamiltonian if the corresponding  $\Theta = X_H^1]\Omega$  is closed  $d\Theta = 0$ , and therefore exact (for a connected space). This allows to find a Hamiltonian  $H$ . In this case, the symplectic form has the form  $\Omega = dx_0 \wedge dx_1$  and  $\Theta = X_H^1]\Omega = dH$ . For example, for the harmonic oscillator,

$$\mathbf{H} = \int H dt = \frac{1}{2} \int (x_0^2 + x_1^2) dt.$$

Poisson structure (1)

$$\{F, G\} = \frac{\partial F}{\partial x_1} \cdot \frac{\partial G}{\partial x_0} - \frac{\partial F}{\partial x_0} \cdot \frac{\partial G}{\partial x_1}$$

is induced by the Hamiltonian vector field

$$X_H^1 = \frac{\partial H}{\partial x_0} \cdot \frac{\partial}{\partial x_1} - \frac{\partial H}{\partial x_1} \cdot \frac{\partial}{\partial x_0},$$

and Cartan distribution (2) defines the dynamical Hamilton equations for  $J^2(\pi) \subset J^1(J^1(\pi))$ :

$$\dot{\mathbf{x}} = \{H, \mathbf{x}\} \quad (3)$$

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<sup>1</sup>E-mail: dumv@comch.ru

in such a way that the volume of Cartan differential forms (2)

$$I = \theta_0 \wedge \theta_1 = \Omega + X_H^1 \rfloor \Omega \wedge dt$$

is an absolute integral invariant and the presymplectic form  $i$  ( $di = I$ )

$$i = \frac{1}{2}(x_0 \wedge dx_1 - x_1 \wedge dx_0) + H \wedge dt$$

gives the Poincaré invariant of the dynamical system in question.

According to Liouville's theorem, any Hamiltonian field preserves the volume form, i.e., the Lie derivative of the 2-form  $\Omega$  with respect to the vector field  $X_H^1$  is zero,  $L_X \Omega = 0$ . In other words, the one-parameter group of symplectic transformations  $\{g_t\}$  (the phase flow) generated by  $X_H^1$  leaves invariant the form  $\Omega$ , i.e.  $g_t^* \Omega = \Omega$ .

**3.** Extending the above discussion to the case of  $J^3(\pi)$ , consider the volume 3-form of the phase space

$$\Omega = dx_0 \wedge dx_1 \wedge dx_2.$$

**Theorem 1.** The volume 3-form  $\Omega \in \Lambda^3$  admits existence of two Hamiltonian polyvector fields  $X_H^1 \in \Lambda^1$  and  $X_H^2 \in \Lambda^2$ .

**Proof.** By definition,

$$L_X \Omega = X \rfloor d\Omega + d(X \rfloor \Omega) = 0.$$

Since  $\Omega \in \Lambda^3$ , we have  $d\Omega = 0$  and

$$d(X \rfloor \Omega) = 0$$

From the Poincaré lemma it follows that the form  $X \rfloor \Omega$  is exact and

$$X \rfloor \Omega = \Theta = d\mathbf{H}.$$

1) If  $X_h^1 \in \Lambda^1$ , then  $\Theta \in \Lambda^2$ ,  $\mathbf{h} = h_i dx_i \in \Lambda^1$ . The Hamiltonian vector field appears as

$$\begin{aligned} X_h^1 &= [\text{rot } h]_i \cdot \frac{\partial}{\partial x_i} \\ &= \left( \frac{\partial h_2}{\partial x_1} - \frac{\partial h_1}{\partial x_2} \right) \frac{\partial}{\partial x_0} + \left( \frac{\partial h_0}{\partial x_2} - \frac{\partial h_2}{\partial x_0} \right) \frac{\partial}{\partial x_1} + \left( \frac{\partial h_1}{\partial x_0} - \frac{\partial h_0}{\partial x_1} \right) \frac{\partial}{\partial x_2}. \end{aligned} \tag{4}$$

Poisson bracket (1) can be written in the form

$$\{\mathbf{h}, G\} = \left( \frac{\partial h_2}{\partial x_1} - \frac{\partial h_1}{\partial x_2} \right) \frac{\partial G}{\partial x_0} + \left( \frac{\partial h_0}{\partial x_2} - \frac{\partial h_2}{\partial x_0} \right) \frac{\partial G}{\partial x_1} + \left( \frac{\partial h_1}{\partial x_0} - \frac{\partial h_0}{\partial x_1} \right) \frac{\partial G}{\partial x_2}.$$

The dynamical equations have the standard form (3):

$$\dot{\mathbf{x}} = \{\mathbf{h}, \mathbf{x}\}$$

2) If  $X_H^2 \in \Lambda^2$ , then  $\Theta \in \Lambda^1$ ,  $H \in \Lambda^0$  and we obtain the Hamiltonian bivector field

$$X_H^2 = \frac{1}{2} \left( \frac{\partial H}{\partial x_0} \cdot \frac{\partial}{\partial x_1} \wedge \frac{\partial}{\partial x_2} + \frac{\partial H}{\partial x_1} \cdot \frac{\partial}{\partial x_2} \wedge \frac{\partial}{\partial x_0} + \frac{\partial H}{\partial x_2} \cdot \frac{\partial}{\partial x_0} \wedge \frac{\partial}{\partial x_1} \right). \quad (5)$$

The Poisson bracket has more complicated expression

$$\begin{aligned} X_H^2 \rfloor (dF \wedge dG) = \{H, F, G\} = & \frac{1}{2} \left[ \frac{\partial H}{\partial x_0} \cdot \left( \frac{\partial F}{\partial x_1} \frac{\partial G}{\partial x_2} - \frac{\partial F}{\partial x_2} \frac{\partial G}{\partial x_1} \right) \right. \\ & \left. + \frac{\partial H}{\partial x_1} \cdot \left( \frac{\partial F}{\partial x_2} \frac{\partial G}{\partial x_0} - \frac{\partial F}{\partial x_0} \frac{\partial G}{\partial x_2} \right) + \frac{\partial H}{\partial x_2} \cdot \left( \frac{\partial F}{\partial x_0} \frac{\partial G}{\partial x_1} - \frac{\partial F}{\partial x_1} \frac{\partial G}{\partial x_0} \right) \right] \end{aligned} \quad (6)$$

and requires introduction of two Hamiltonians for the dynamical equation

$$\dot{\mathbf{x}} = \{H, F, \mathbf{x}\}. \quad (7)$$

The use of a vector Hamiltonian allows ones to obtain a generalization of the Poincaré invariants. Since the volume of the Cartan distribution

$$\theta = d\mathbf{x} - \{H, \mathbf{x}\} \wedge dt$$

is the absolute integral invariant  $I = \theta_0 \wedge \theta_1 \wedge \theta_2$ , or

$$I = \Omega - X^1 \rfloor \Omega \wedge dt,$$

the pretrisymplectic form  $i \quad (I = di)$

$$i = \omega - (h_i dx_i) \wedge dt$$

gives the Poincaré invariant of the dynamical system in question. Here

$$\omega = x_0 dx_1 \wedge dx_2 + x_1 dx_0 \wedge dx_2 + x_2 dx_0 \wedge dx_1$$

is the Kirillov symplectic form ( $\Omega = d\omega$ ).

**Example 1.** Consider the equations of motion of a "solid body"

$$\begin{cases} \dot{x} = y - z \\ \dot{y} = z - x \\ \dot{z} = x - y \end{cases}$$

The Lax pair

$$\dot{L} = [ML], \quad L = \begin{pmatrix} x & z & y \\ z & y & x \\ y & x & z \end{pmatrix}, \quad M = \frac{1}{2} \begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{pmatrix}$$

for this flow gives two invariants

$$I_1 = \text{tr } L = x + y + z, \quad I_2 = \frac{1}{2} \text{tr } L^2 = \frac{3}{2}(x^2 + y^2 + z^2),$$

and Poisson-Nambu [3] structure (7)

$$\dot{x}_i = \{H, F, x_i\} = X_H \rfloor dF \wedge dx_i = -X_F \rfloor dH \wedge dx_i,$$

is determined by vector field (6)

$$\begin{aligned} X_H &= z \frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y} + x \frac{\partial}{\partial y} \wedge \frac{\partial}{\partial z} + y \frac{\partial}{\partial z} \wedge \frac{\partial}{\partial x}, \\ X_F &= \frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y} + \frac{\partial}{\partial y} \wedge \frac{\partial}{\partial z} + \frac{\partial}{\partial z} \wedge \frac{\partial}{\partial x} \end{aligned}$$

with two Hamiltonians  $H = \frac{1}{3}I_2$ ,  $F = I_1$ .

To determine the Hamiltonian vector field, we rewrite the equation of motion of a solid body in the form

$$\dot{\mathbf{x}} = J\mathbf{x}, \quad J = -2M.$$

This vector flow is called Hamiltonian if

$$\text{div } J\mathbf{x} = 0.$$

This may mean that

$$J\mathbf{x} = \text{rot } h.$$

The latter expression is the requirement that some differential form is closed  $d\omega = 0$  :

$$\omega = (y - x)dy \wedge dz + (-x + z)dz \wedge dx + (x - y)dx \wedge dy.$$

Using the homotopy operator, we obtain

$$\omega = d\mathbf{h}, \quad \mathbf{h} = (h \cdot dx),$$

where

$$h = \frac{1}{3} \begin{pmatrix} y^2 + z^2 - x(y + z) \\ z^2 + x^2 - y(z + x) \\ x^2 + y^2 - z(x + y) \end{pmatrix}, \quad dx = \begin{pmatrix} dx \\ dy \\ dz \end{pmatrix}.$$

Now the Hamiltonian vector field takes the form

$$\begin{aligned}
X_h^1 &= \left( \text{rot } \mathbf{h} \cdot \frac{\partial}{\partial \mathbf{x}} \right) \\
&= \left( \frac{\partial h_3}{\partial y} - \frac{\partial h_2}{\partial z} \right) \frac{\partial}{\partial x} + \left( \frac{\partial h_1}{\partial z} - \frac{\partial h_3}{\partial x} \right) \frac{\partial}{\partial y} + \left( \frac{\partial h_2}{\partial x} - \frac{\partial h_1}{\partial y} \right) \frac{\partial}{\partial z} \\
&= \frac{1}{3}(y^2 + z^2 - x(y + z)) \frac{\partial}{\partial x} \\
&\quad + \frac{1}{3}(z^2 + x^2 - y(z + x)) \frac{\partial}{\partial y} + \frac{1}{3}(x^2 + y^2 - z(x + y)) \frac{\partial}{\partial z},
\end{aligned}$$

and the Poisson bracket  $\{\mathbf{h}, G\} = X_h \rfloor dG$  gives

$$\dot{x}_i = \{\mathbf{h}, x_i\} = X_h \rfloor dx_i, \quad (8)$$

where  $\mathbf{h}$  is the vector Hamiltonian and

$$d\mathbf{h} = dH \wedge dF.$$

**Example 2.** The Ishii equation [2]  $\ddot{x} = \dot{x}x \in J^3(1, 1)$  can be rewritten in the form of a flow in  $J^1(1, 3)$ :

$$\dot{\mathbf{x}} = J\mathbf{x}, \quad J = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ y & 0 & 0 \end{pmatrix}.$$

The Lax representation for this flow

$$\dot{L} = [ML], \quad L = \begin{pmatrix} -x & 0 & 1 \\ -y & 0 & 0 \\ -z & \frac{y}{2} & 0 \end{pmatrix}, \quad M = \frac{1}{2} \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 2 \\ y & x & 2 \end{pmatrix}$$

gives two invariants

$$I_2 = \frac{1}{2} \text{tr} L^2 = \frac{x^2}{2} - z, \quad I_1 = \frac{1}{3} \text{tr} L^3 = xz - \frac{y^2}{2} - \frac{x^3}{3}$$

for Eq.(7). The vector Hamiltonian appears as

$$\mathbf{h} = \frac{1}{4} \begin{pmatrix} z^2 - xy^2 \\ x^2y - 2yz \\ y^2 - 2xz \end{pmatrix}$$

These Hamiltonians are connected by expressions

$$d\mathbf{h} = dI_1 \wedge dI_2.$$

This means that our system admit Poisson structure with vectorial Hamiltonian

$$\dot{x}_i = \{\mathbf{h}, x_i\} = X_h \rfloor dx_i,$$

where

$$X_h = y \frac{\partial}{\partial x} + z \frac{\partial}{\partial y} + xy \frac{\partial}{\partial z},$$

and Poisson-Nambu structure

$$\dot{x}_i = \{I_1, I_2, x_i\} = X_{I_1} \rfloor dI_2 \wedge dx_i = -X_{I_2} \rfloor dI_1 \wedge dx_i,$$

where

$$\begin{aligned} X_{I_1} &= x \frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y} + (z - x^2) \frac{\partial}{\partial y} \wedge \frac{\partial}{\partial z} - y \frac{\partial}{\partial z} \wedge \frac{\partial}{\partial x}, \\ X_{I_2} &= x \frac{\partial}{\partial y} \wedge \frac{\partial}{\partial z} - \frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y}. \end{aligned}$$

**4.** In the generalization of the above discussion to the case  $J^n(\pi)$  we will rely upon Liouville's theorem. Namely, we will seek for polyvector fields preserving the multisymplectic volume  $n$ -form

$$\Omega = dx_0 \wedge dx_1 \wedge \dots \wedge dx_{n-1}$$

of the phase space.

**Theorem 2.** The multisymplectic volume  $n$ -form of the phase space  $\Omega \in \Lambda^n$  admits existence of  $n - 1$  Hamiltonian polyvector fields  $X^k \in \Lambda^k$ .

**Proof** of this theorem is similar to the proof of Theorem 1. It should only be noted that if  $\mathbf{X}_H^k \in \Lambda^k$ , then  $\Theta \in \Lambda^{n-k}$ ,  $H \in \Lambda^{n-k-1}$ . In order for  $H \in \Lambda^m$  ( $m \geq 0$ ), it is necessary that ( $k \leq n - 1$ ).

A peculiarity of this problem is that  $\Omega \in \Lambda^n$ , where  $\Lambda(R^n) \supset (\Lambda^0, \Lambda^1, \dots, \Lambda^n)$  is a graded exterior algebra. If the phase space is connected, a closed form  $\Theta$  is exact. But the quotient algebra  $\Lambda(R^n)/\Lambda^n$  admits existence of  $(n - 1)$  exact forms  $\Theta \in \Lambda^{m=n-k}$  ( $1 \leq m \leq n - 1$ ), depending on whether  $X^k \in \Lambda^k$  ( $n - 1 \geq k \geq 1$ ) or not.

It is not difficult to present several Hamiltonian polyvector fields

$$\begin{aligned} X_H^{n-1} &= \frac{1}{(n-1)!} \sum_{k=0}^{n-1} \frac{\partial H}{\partial x_k} \cdot \frac{\partial}{\partial x_0} \wedge \frac{\partial}{\partial x_1} \wedge \left[ \frac{\partial}{\partial x_k} \right] \wedge \dots \wedge \frac{\partial}{\partial x_{n-1}}, \\ X_H^{n-2} &= \frac{1}{(n-2)!} \sum_{i < k}^{n-1} \left( \frac{\partial H}{\partial x_i} - \frac{\partial H}{\partial x_k} \right) \cdot \frac{\partial}{\partial x_0} \wedge \left[ \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_k} \right] \wedge \dots \wedge \frac{\partial}{\partial x_{n-1}}, \\ X_H^{n-3} &= \frac{1}{(n-3)!} \sum_{i < k < l}^{n-1} \left( \frac{\partial H}{\partial x_i} - \frac{\partial H}{\partial x_k} + \frac{\partial H}{\partial x_l} \right) \cdot \left[ \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_k} \wedge \frac{\partial}{\partial x_l} \right] \wedge \dots \wedge \frac{\partial}{\partial x_{n-1}}, \end{aligned}$$

... ..

for which

$$\begin{aligned}
\Theta^1 &= X_H^{n-1} \rfloor \Omega = dH, \\
\Theta^2 &= X_H^{n-2} \rfloor \Omega = \sum_i^{n-1} dH \wedge dx_i, \\
\Theta^3 &= X_H^{n-3} \rfloor \Omega = \sum_{i < j}^{n-1} dH \wedge dx_i \wedge dx_j, \\
\Theta^4 &= X_H^{n-4} \rfloor \Omega = \sum_{i < j < k}^{n-1} dH \wedge dx_i \wedge dx_j \wedge dx_k, \\
&\dots \dots \dots \\
\Theta^{n-1} &= X_H^1 \rfloor \Omega = \sum_{i < j < \dots < k}^{n-1} dH \wedge dx_i \wedge dx_j \wedge \dots \wedge dx_k.
\end{aligned}$$

For any  $n$ -form  $\Omega$ , the corresponding Hamiltonian vector field  $X_H^1 \in \Lambda^1$  generates the Poisson structure  $X_H^1 \rfloor dx_i = \{H, x_i\}$  and the contact vector field

$$X_H = \frac{\partial}{\partial t} + X_H^1$$

of distribution(2).

**5.** The Nambu theory [3] supposes introduction  $(n - 1)$  Hamiltonians for description of the  $n$ -dimensional phase space  $\Omega$ . The use of Hamiltonian polyvector fields leads to the following generalization of this theory.

**Theorem 3.** The Poisson structure of an  $n$ -dimensional multisymplectic space  $\Omega$  is induced by  $(n - 1)$  Hamiltonian polyvector fields  $X_H^k \in \Lambda^k$  each of which requires introduction of  $k$  Hamiltonians:

$$X_H^k \rfloor (dF_1 \wedge dF_2 \wedge \dots \wedge dF_k) = \{F_1, F_2, \dots, F_k, H\}.$$

**Example 3.** This example can be of interest for the description of multilevel systems of the quantum information theory. In the capacity of carriers of quantum information, at present, observable generators of the group  $SO(3)$  (isomorphic to  $SU(2)$ ) are considered. To obtain irreducible representations of the group  $SU(2)$ , the standard method of constructing proper states of the angular momentum operator is applied. In addition, the state indication operator

$$\Sigma_3 |n, m\rangle = m |n, m\rangle$$

and the creation-destruction operators (level increasing or decreasing operators) with boundary properties

$$\Sigma_+ |n, n\rangle = \Sigma_- |n, -n\rangle = 0$$

are considered.

Another way ([4].P.8) is to use annular  $n$ -level systems with periodicity condition

$$|n, i + n\rangle = |n, i\rangle.$$

This allows ones to use in the description the two operators

$$\Sigma_1 |n, i\rangle = |n, i + 1\rangle \pmod{n}, \quad \Sigma_3 |n, i\rangle = i |n, i\rangle,$$

where

$$\Sigma_1 = \begin{pmatrix} 0 & & & 1 \\ 1 & 0 & & \\ & 1 & \ddots & \\ & & \ddots & 0 \\ & & & 1 & 0 \end{pmatrix}, \quad \Sigma_3 = \begin{pmatrix} \sigma^0 & & & \\ & \sigma^1 & & \\ & & \sigma^2 & \\ & & & \ddots \\ & & & & \sigma^n \end{pmatrix},$$

Here  $\sigma = e^{i\frac{2\pi}{n}}$  has the following properties:

$$\sigma^n = 1, \quad \sigma^{n+k} = \sigma^k \pmod{n}, \quad \sigma = \sigma^{n+1}, \quad \sum_{k=0}^n \sigma^k = 0.$$

In what follows the generators  $\Sigma_1$  and  $\Sigma_3$  will be called the generalized Pauli matrices. We will denote the identity matrix by  $\mathbf{I} = \Sigma_0$  and the increasing operator by

$$\Sigma_1^+ = \Sigma_1^{n-1} = \begin{pmatrix} 0 & 1 & & \\ & 0 & 1 & \\ & & \ddots & \\ & & & 0 & 1 \\ 1 & & & & 0 \end{pmatrix}.$$

In particular, for  $n = 2$ , the expansion of the operator  $e^{t\Sigma_1}$  into a series gives

$$e^{t\Sigma_1} = c_0(t)\Sigma_0 + c_1(t)\Sigma_1,$$

where

$$c_0(t) = \sum_{k=0}^{\infty} \frac{t^{2k}}{(2k)!}, \quad c_1(t) = \sum_{k=0}^{\infty} \frac{t^{2k+1}}{(2k+1)!} \quad (9)$$

In this case, we let

$$c_0(t) = \cosh(t), \quad c_1(t) = \sinh(t)$$



and, taking into account that

$$\frac{d}{dt} \cosh(t) = \sinh(t), \quad \frac{d}{dt} \sinh(t) = \cosh(t)$$

obtain the system of differential equations

$$\dot{\mathbf{c}} = \Sigma_1^+ \mathbf{c}, \quad (10)$$

where  $\mathbf{c}(t)$  is the vector with coordinates  $\mathbf{c} = (c_0, c_1)$ . The Lax representation appears as

$$L = \begin{pmatrix} c_1 y & c_0 \\ c_0 \sigma^1 & c_1 \sigma^1 \end{pmatrix}, \quad M = \frac{\sigma^1}{2} \Sigma_1$$

It gives the scalar invariant

$$I = \frac{1}{2} \text{tr} L^2 = c_1^2 - c_0^2$$

for Poisson bracket (3). Note that the relation  $I = 1$  is the basic trigonometric identity for functions (9).

For  $n = 3$ , the expansion of the operator  $e^{t\Sigma_1}$  into a series gives

$$e^{t\Sigma_1} = c_0(t)\Sigma_0 + c_1(t)\Sigma_1 + c_2(t)\Sigma_1^2,$$

where

$$c_0(t) = \sum_{k=0}^{\infty} \frac{t^{3k}}{(3k)!}, \quad c_1(t) = \sum_{k=0}^{\infty} \frac{t^{3k+1}}{(3k+1)!}, \quad c_2(t) = \sum_{k=0}^{\infty} \frac{t^{3k+2}}{(3k+2)!}. \quad (11)$$

The obtained coefficients again satisfy system of differential equations (10), which has the following form in the Lax representation:

$$L = \begin{pmatrix} c_0 \sigma^2 & c_1 \sigma^1 & c_2 \sigma^0 \\ c_2 \sigma^1 & c_0 \sigma^0 & c_1 \sigma^2 \\ c_1 \sigma^0 & c_2 \sigma^2 & c_0 \sigma^1 \end{pmatrix}, \quad M = \frac{1}{\sigma^1 - \sigma^2} \Sigma_1.$$

This system gives the only one scalar invariant

$$I = \frac{1}{3} \text{tr} L^3 = c_0^3 + c_1^3 + c_2^3 - 3c_0 c_1 c_2.$$

Note that the expression  $I = 1$  is the basic trigonometric identity for functions (11). The vector Hamiltonian for bracket (8) appears as

$$\mathbf{h} = \frac{1}{4} \begin{pmatrix} c_2^2 - 2c_0 c_1 \\ c_0^2 - 2c_1 c_2 \\ c_1^2 - 2c_0 c_2 \end{pmatrix}.$$

In the general case, we have

$$e^{t\Sigma_1} = c_0(t)\Sigma_0 + c_1(t)\Sigma_1 + \dots + c_{n-2}(t)\Sigma_1^{n-2} + c_{n-1}(t)\Sigma_1^{n-1},$$

where the functions

$$c_j(t) = \sum_{k=0}^{\infty} \frac{t^{nk+j}}{(nk+j)!}$$

satisfy system (10) with Lax pair

$$L = \begin{pmatrix} c_0\sigma^{n-1} & c_1\sigma^{n-2} & c_2\sigma^{n-3} & \dots & c_{n-3}\sigma^2 & c_{n-2}\sigma^1 & c_{n-1}\sigma^0 \\ c_1\sigma^{n-2} & c_0\sigma^{n-3} & c_1\sigma^{n-4} & \dots & c_{n-4}\sigma^1 & c_{n-3}\sigma^0 & c_{n-2}\sigma^{n-1} \\ c_2\sigma^{n-3} & c_3\sigma^{n-4} & c_0\sigma^{n-5} & \dots & c_{n-5}\sigma^0 & c_{n-4}\sigma^{n-1} & c_{n-3}\sigma^{n-2} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ c_3\sigma^2 & c_4\sigma^1 & c_{n-1}\sigma^0 & \dots & c_0\sigma^5 & c_1\sigma^4 & c_2\sigma^3 \\ c_2\sigma^1 & c_3\sigma^0 & c_4\sigma^{n-1} & \dots & c_{n-1}\sigma^4 & c_0\sigma^3 & c_1\sigma^2 \\ c_1\sigma^0 & c_2\sigma^{n-1} & c_3\sigma^{n-2} & \dots & c_{n-2}\sigma^3 & c_{n-1}\sigma^2 & c_0\sigma^1 \end{pmatrix},$$

$$M = \frac{1}{\sigma^1 - \sigma^{n-1}} \Sigma_1.$$

The vector invariant has a rather complicated form. For example, for  $n = 4$ , we have the scalar invariant

$$I = 2(2c_0c_2 - c_1^2 - c_4^2),$$

and the differential form of vector Hamiltonian

$$\begin{aligned} \mathbf{h} = (h \cdot dc) &= \frac{1}{6} [(c_3^2 - 2c_0c_2)dc_0 \wedge dc_1 + (c_1^2 - 2c_2c_0)dc_2 \wedge dc_3 \\ &+ (c_2^2 - 2c_3c_1)dc_0 \wedge dc_3 + (c_0^2 - 2c_3c_1)dc_2 \wedge dc_1 \\ &+ 2(c_3c_0 - c_1c_2)dc_1 \wedge dc_3 + 2(c_0c_1 - c_2c_3)dc_0 \wedge dc_2]. \end{aligned}$$

In all the cases, the invariants are in involution, and the Poisson bracket  $\{\mathbf{h}, I\} = 0$  gives no new integrals.

## References

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